

CONSTRUCTION OF A SPECTRALLY STABLE SELF-SIMILAR BLOWUP SOLUTION TO THE SUPERCRITICAL HARMONIC MAP HEAT FLOW

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ABSTRACT. We prove the existence of a (spectrally) stable self-similar blow-up solution f_0 to the heat flow for corotational harmonic maps from \mathbb{R}^3 to the three-sphere. In particular, our result verifies the spectral gap conjecture stated by one of the authors and lays the groundwork for the proof of the nonlinear stability of f_0 . At the heart of our analysis lies a new existence result of a monotone self-similar solution f_0 . Although solutions of this kind have already been constructed before, our approach reveals substantial quantitative properties of f_0 , leading to the stability result. A key ingredient is the use of interval arithmetic: a rigorous computer-assisted method for estimating functions.

1. INTRODUCTION

Let (M, g) and (N, h) be two Riemannian manifolds with metrics g and h , respectively. *Harmonic maps* $F : M \rightarrow N$ are defined as critical points of the functional¹

$$S(F) = \int_M g^{jk} \partial_j F^a \partial_k F^b h_{ab} \circ F,$$

which is a generalization of the classical Dirichlet energy. Harmonic maps have a number of applications in physics, e.g. in the description of ferromagnetism. Given two manifolds M and N , a natural mathematical problem is to construct or, ideally, characterize harmonic maps from M to N . A classical device for that purpose is the associated *harmonic map heat flow* which may be used to deform an arbitrary map to a harmonic one [11]. This works well under certain assumptions on the curvature but in general fails due to the onset of singularities in finite time. The goal is then to develop a sufficiently good understanding of singularity formation in order to continue the flow beyond the singularity in a suitable manner. To this end it is necessary to understand the *generic* blowup behavior of the flow.

In many cases it is possible to demonstrate finite-time blowup by constructing explicit solutions. However, the relevance of these particular examples with respect to generic behavior is strongly dependent on their stability. The aim of this paper is to study stable singularities of the harmonic map heat flow in the case $M = N = \mathbb{S}^3$. As a matter of fact, the curvature of the base manifold M is irrelevant for the asymptotic behavior near a singularity and thus, for simplicity, we may equally well set $M = \mathbb{R}^3$. Furthermore, we restrict ourselves to *corotational maps* $F : \mathbb{R}^3 \rightarrow \mathbb{S}^3$ which are of the form $F(r, \theta, \varphi) = (u(r), \theta, \varphi)$, where (r, θ, φ)

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¹Einstein's summation convention is in force.

are the standard spherical coordinates on \mathbb{R}^3 , and we use hyperspherical coordinates on \mathbb{S}^3 . The heat flow for such maps is then described by the parabolic Cauchy problem

$$\begin{cases} \partial_t u(r, t) = \partial_r^2 u(r, t) + \frac{2}{r} \partial_r u(r, t) - \frac{\sin(2u(r, t))}{r^2}, \\ u(r, 0) = rv_0(r), \quad \|v_0\|_\infty < \infty, \end{cases} \quad (1.1)$$

where now $u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is time-dependent. An analogous symmetry reduction is possible for maps $F : \mathbb{R}^d \rightarrow \mathbb{S}^d$.

As in [18, 1, 22, 7, 19], we focus on the blow-up scenario, where a solution $u(r, t)$, starting from smooth initial data, develops a rapidly increasing gradient at $r = 0$,

$$\lim_{t \rightarrow T^-} \partial_r u(0, t) = \infty.$$

Here, $T > 0$ is the blow-up time. One finds that in dimensions $d = 3, 4, 5, 6$ the gradient increases according to the parabolic scaling symmetry of the equation ($r \rightarrow \lambda r$ and $T - t \rightarrow \lambda^2(T - t)$), so that $\partial_r u(0, t) \propto (T - t)^{-\frac{1}{2}}$. This is referred to as self-similar blow-up and has been studied numerically [3] and rigorously [13, 12]. In higher dimensions $d > 7$, the blow-up takes a more complicated form as described in [2, 5, 6].

The harmonic map heat flow bares a striking similarity to other parabolic equations, also displaying a blow-up scenario, such as Yang-Mills flow and semilinear heat equation. In fact, one of the authors and Schörkhuber have recently proved the nonlinear stability of a self-similar solution for the Yang-Mills flow [10]. The proof in [10] relies on a closed-form expression for the self-similar profile to solve the spectral stability problem. Such a closed-form expression is unavailable for the harmonic map flow but in this paper we show how to circumvent this issue. Similar approaches were used in [9, 8], which inspired this paper.

As already noted, the Cauchy problem (1.1) has long been known [12, 13] to possess self-similar solutions of the form

$$u(r, t) = f\left(\frac{r}{\sqrt{T - t}}\right),$$

where f solves the boundary value problem

$$f''(y) + \left(\frac{2}{y} - \frac{y}{2}\right) f'(y) - \frac{1}{y^2} \sin(2f(y)) = 0, \quad y \geq 0, \quad f(0) = 0, \quad f(\infty) = \text{const.} \quad (1.2)$$

In [12], it is proved that there exists a countable family of solutions, denoted by $\{f_n\}_{n=0,1,\dots}$, indexed by their number of intersections with $\pi/2$. Each solution f_n is shown to have $n - 1$ extrema and n intersections with $\pi/2$. In [13], one finds a related existence result: it is proved that there exists a monotone solution to (1.2), which crosses $\pi/2$ exactly once. In addition to the rigorous results, a family of self-similar solutions, with the same qualitative properties as the ones from [12], was found numerically in [3]. On top of the existing results, our paper adds yet another proof of existence of a monotone self-similar solution.

Theorem 1.1. *Let*

$$\tilde{f}_0(y) := 2 \arctan \left(\sum_{n=0}^{14} (f_0)_n T_{2n+1} \left(\frac{y}{\sqrt{2 + y^2}} \right) \right),$$

with coefficients $(f_0)_n$ given in Table B.1 and T_n being the standard Chebyshev polynomials. There exists a monotone solution $f_0 \in C^\infty([0, \infty))$ to (1.2) such that

$$\|f_0 - \tilde{f}_0\| \leq 5 \cdot 10^{-4} \quad (1.3)$$

where the norm $\|\cdot\|$ is given by

$$\|f\| := \|p_1 f\|_{L^\infty(0, \infty)} + \|p_3 f'\|_{L^\infty(0, \infty)}, \quad p_1(y) = \frac{\sqrt{2+y^2}}{\sqrt{2}y}, \quad p_3(y) = \frac{(2+y^2)^{3/2}}{2\sqrt{2}}.$$

Remark 1.2. Interestingly, there is still no uniqueness result for self-similar solutions: we do not know if the family of solutions found in [12] is exhaustive. Even worse, strictly speaking we do not know if the solutions f_0 found in this and other papers [13, 12] are the same (which, however, is very reasonable to assume). Hopefully, a result similar to [17] can be established in the future. From now on, to avoid confusion, whenever we refer to f_0 we mean the solution from Theorem 1.1.

In addition to the existence result we have the following technical proposition to describe some qualitative properties of f_0 needed in the proof of nonlinear stability in [4].

Proposition 1.3. *Any solution $f \in C^\infty([0, \infty))$ to (1.2) has vanishing even derivatives at $y = 0$, that is,*

$$f^{(2k)}(0) = 0, \quad k \in \mathbb{N}_0,$$

and for each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$|f^{(k)}(y)| \leq C_k y^{-2-k}$$

for all $y \geq 1$.

Although Theorem 1.1 seems superfluous at first glance (we already mentioned two other proofs finding similar solutions), our approach represents a significant advantage over the previous ones: the explicit form of \tilde{f}_0 allows us to rigorously show that our solution is *spectrally stable* (modulo a gauge mode), see Theorem 1.6 below for the precise meaning of this. Indeed, the linear stability of a monotone solution was conjectured in [3], where it is claimed that a (self-adjoint) linear operator associated to it has no unstable eigenvalues (i.e. eigenvalues in the interval $(-\infty, 0]$), apart from a gauge eigenvalue $\lambda = -1$ ². The latter refers to an eigenvalue that is related to the time translation symmetry of Eq. (1.1) and which does not constitute a “real” instability. The claim of linear stability in [3] was supported by a simple Sturm oscillation argument, which excludes eigenvalues $\lambda \in (-\infty, -1)$, along with a numerical test to exclude $\lambda \in (-1, 0]$. The Sturm oscillation argument can be made rigorous very easily (and in fact we use it here), but the rigorous exclusion of eigenvalues in the interval $(-1, 0]$ seems impossible without additional quantitative information on the profile of the self-similar solution. Our main motivation for establishing Theorem 1.1 in its particular form is to provide this missing quantitative information.

²One can find an even stronger conjecture in [3], namely that f_0 is a generic solution attractor for a large set of data. Our result serves as the first step in proving this conjecture.

To analyze the stability of a self-similar solution f_0 , let us consider equation (1.1) in self-similar variables

$$\begin{aligned} s &= -\log(T-t), & y &= \frac{r}{\sqrt{T-t}}, & f(y, s) &= u(r, t), \\ \partial_s f &= \partial_y^2 f + \left(\frac{2}{y} - \frac{y}{2}\right) \partial_y f - \frac{1}{y^2} \sin(2f). \end{aligned} \tag{1.4}$$

Evidently, f_0 , being a solution to (1.2), automatically leads to a stationary solution $f(y, s) = f_0(y)$ to (1.4) and thus a solution global in time $s = -\log(T-t)$. Let us set $f(y, s) = f_0(y) + e^{-\lambda s} w(y)$ and linearize in w , to get the spectral problem $(\lambda - \mathcal{A}_0)w = 0$ for the operator

$$\begin{aligned} \mathcal{A}_0 w(y) &= -\frac{1}{\rho(y)} \partial_y [\rho(y) \partial_y w(y)] + \frac{2 \cos(2f_0(y))}{y^2} w(y) \\ &= -\frac{1}{\rho(y)} \partial_y [\rho(y) \partial_y w(y)] + \frac{2}{y^2} w(y) + V_0(y) w(y) \\ \rho(y) &= y^2 e^{-\frac{y^2}{4}}, & V_0(y) &= \frac{-4 \sin(f_0(y))^2}{y^2}. \end{aligned} \tag{1.5}$$

Consider \mathcal{A}_0 as an operator on the weighted L^2 -space

$$\mathcal{H} := L^2_\rho(0, \infty)$$

with domain $C_0^\infty(0, \infty)$. In this setting, \mathcal{A}_0 is densely defined and symmetric. It is not hard to see that the possible endpoint behavior of solutions to $\mathcal{A}_0 w = 0$ at 0 is $w(y) \propto y$ and $w(y) \propto y^{-2}$. Consequently, only the recessive solution belongs to \mathcal{H} and thus, by the Weyl alternative, \mathcal{A}_0 is in the limit-point case at 0. Similarly, close to ∞ we have the two behaviors $w(y) \propto 1$ and $w(y) \propto y^{-1} e^{\frac{y^2}{4}}$ implying that \mathcal{A}_0 is limit-point at infinity, too. By Theorem X.7 from [20], we conclude that \mathcal{A}_0 is essentially self-adjoint (the theorem in [20] applies to operators of the form $-\frac{d^2}{dx^2} + V(x)$ but it is easy enough to reduce \mathcal{A}_0 to that form by changing variables to $v(y) = \rho(y)^{\frac{1}{2}} w(y)$). Furthermore, from the endpoint behavior it follows that the closure of \mathcal{A}_0 has compact resolvent. In summary, we arrive at the following basic result on the spectral theory of \mathcal{A}_0 .

Proposition 1.4. *The operator \mathcal{A}_0 is essentially self-adjoint and the spectrum of its closure consists of a countable number of real, simple eigenvalues.*

Definition 1.5. From now on we will denote by \mathcal{A}_0 the unique self-adjoint extension on $L^2_\rho(0, \infty)$ of the formal differential operator defined in (1.5).

From the perturbation ansatz $f(y, s) = f_0(y) + e^{-\lambda s} w(y)$ it is evident that negative eigenvalues of \mathcal{A}_0 lead to linear instabilities of f_0 . As a matter of fact, there exists the negative eigenvalue -1 but this is a gauge eigenvalue, i.e., it is related to the freedom of choosing the parameter T in the definition of the self-similar variables and therefore it is not related to an instability of f_0 . Consequently, to prove linear stability of f_0 it is necessary to rule out the existence of negative eigenvalues of \mathcal{A}_0 other than -1 . As already mentioned, the most difficult part is to prove the absence of eigenvalues in $(-1, 0]$ since this hinges on the particular shape of the potential V_0 . Thus, a rigorous proof of this *spectral gap property* requires quantitative information on f_0 . Our estimates (1.3) on f_0 lead to very precise bounds on V_0

and in turn allow us to prove the following stability result. This result is an indispensable ingredient in the proof of the nonlinear asymptotic stability of f_0 in the companion paper [4].

Theorem 1.6. *The only eigenvalue of the operator \mathcal{A}_0 in the interval $(-\infty, 0]$ is -1 .*

Furthermore, the strict bounds on f_0 could potentially lead to a rigorous treatment of continuation beyond the blow-up, along the lines of the following informal reasoning. In [3] it is conjectured that one can construct a unique continuation for a solution

$$u_0(r, t) := f_0\left(\frac{r}{\sqrt{T-t}}\right), \quad t < T$$

past the blow-up time by defining

$$u_0(r, t) := g_0\left(\frac{r}{\sqrt{t-T}}\right), \quad t > T.$$

In the above definition, g_0 is a profile of an expanding self-similar solution satisfying

$$g''(y) + \left(\frac{2}{y} + \frac{y}{2}\right) g'(y) - \frac{1}{y^2} \sin(2g(y)) = 0, \quad g(0) \in \{0, \pi\}, \quad g(\infty) = \text{const.} \quad (1.6)$$

Note that in the above boundary value problem the expanding profile g can freely select the boundary condition at $y = 0$, which means that $u_0(0, t)$ may jump from 0 to π . In effect, the underlying map $F : \mathbb{R}^3 \rightarrow \mathbb{S}^3$ would change its homotopy class in consequence of the blow-up. One of the main motivations for studying the blow-up patterns is to determine whether such a jump occurs.

The formal construction in [3] requires that g_0 satisfies the following matching condition

$$g_0(\infty) = \lim_{t \rightarrow T+} u_0(r, t) = \lim_{t \rightarrow T-} u_0(r, t) = f_0(\infty). \quad (1.7)$$

That is, the asymptotics of g_0 and f_0 have to coincide at infinity. In this sense, the question of unique continuation past the blow-up can be reduced to the question of uniqueness of solutions to the boundary value problem (1.6) with a boundary condition $g_0(\infty) = f_0(\infty)$. Germain and Rupflin, who studied the boundary value problem (1.6) in [15], prove that the closer $g(\infty)$ is to $\pi/2$ the more solutions to (1.6) there are. In the more recent paper [14] it is shown that there are at least two stable expanding self-similar solutions with the same initial data, provided $g(\infty)$ is sufficiently close to $\pi/2$. For f_0 from Theorem 1.1 one can explicitly compute that $|f_0(\infty) - \pi/2| > 0.56$ which is conjecturally large enough to allow for a unique continuation.

2. EXISTENCE OF A SELF-SIMILAR SOLUTION

The standard approach would be to define $\delta = f - \tilde{f}_0$ and rewrite the equation (1.2) as

$$\mathcal{L}\delta = \mathcal{R}(\tilde{f}_0) + \mathcal{N}(\delta) \quad (2.1)$$

with

$$\begin{aligned}
\mathcal{L}\delta(y) &= -\delta''(y) - \left(\frac{2}{y} - \frac{y}{2}\right) \delta'(y) + \frac{2}{y^2} \delta(y) + \tilde{V}_0(y) \delta(y), \quad \tilde{V}_0(y) = \frac{-4 \sin(\tilde{f}_0(y))^2}{y^2} \\
\mathcal{R}(\tilde{f}_0)(y) &= \tilde{f}_0''(y) + \left(\frac{2}{y} - \frac{y}{2}\right) \tilde{f}_0'(y) - \frac{1}{y^2} \sin(2\tilde{f}_0(y)) \\
\mathcal{N}(\delta)(y) &= -\frac{1}{y^2} \left(\sin(2\tilde{f}_0(y) + 2\delta(y)) - \sin(2\tilde{f}_0(y)) - 2\cos(2\tilde{f}_0(y))\delta(y) \right) \\
&= \frac{2}{y^2} \left(\cos(2\tilde{f}_0(y))\delta(y) - \cos(2\tilde{f}_0(y) + \delta(y)) \sin(\delta(y)) \right).
\end{aligned}$$

The goal now is to invert the operator \mathcal{L} and prove that

$$\mathcal{K}(\delta) := \mathcal{L}^{-1}(\mathcal{R}(\tilde{f}_0) + \mathcal{N}(\delta))$$

is a contraction if $\|\delta\|$ is small enough. Unfortunately this plan cannot succeed as the operator \mathcal{L} contains a very complicated potential \tilde{V}_0 : a nonlinear function of \tilde{f}_0 , which itself is already complicated. Because of the complicated form of the potential, \mathcal{L} cannot be inverted explicitly.

The remedy comes in the form of the following trick. Imagine we can construct an operator $\tilde{\mathcal{L}}$, which we can invert explicitly, and we rewrite Eq. (2.1) as

$$\tilde{\mathcal{L}}\delta = \mathcal{R}(\tilde{f}_0) + \mathcal{N}(\delta) + (\tilde{\mathcal{L}} - \mathcal{L})\delta. \quad (2.2)$$

If, in addition, the difference $\tilde{\mathcal{L}} - \mathcal{L}$ is small in a suitable sense, then the map

$$\tilde{\mathcal{K}}(\delta) := \tilde{\mathcal{L}}^{-1}(\mathcal{R}(\tilde{f}_0) + \mathcal{N}(\delta) + (\tilde{\mathcal{L}} - \mathcal{L})\delta)$$

turns out to be the right object to apply a contraction mapping principle to, as we can estimate all the objects on the right hand side explicitly. Because solving (2.2) is equivalent to solving (2.1), it is sufficient to find a fixed point of $\tilde{\mathcal{K}}$. At this point, it remains to show that $\tilde{\mathcal{K}}$ is a contraction on a suitable closed subset of a Banach space, chosen here as

$$X = \{\delta \in C^1([0, \infty)), \|\delta\| \leq 5 \cdot 10^{-4}\}$$

with $\|\cdot\|$ given in Theorem 1.1. The rest of the proof is divided as follows, where we abbreviate $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(0, \infty)}$.

(1) We construct an operator $\tilde{\mathcal{L}}$ so that

$$\|\tilde{\mathcal{L}}^{-1}\alpha\| \leq c_{\mathcal{L}}\|p_2\alpha\|_\infty \quad p_2(y) = \frac{(2+y^2)^{5/2}}{3\sqrt{2}y(4+y^2)},$$

for $c_{\mathcal{L}} = 180$ and any function α with a finite $\|p_2\alpha\|_\infty$. In addition, for the constructed operator $\tilde{\mathcal{L}}$, the difference $\tilde{\mathcal{L}} - \mathcal{L}$ is small (see the estimate in the next point).

(2) We show that for $\delta, \gamma \in X$

$$\begin{aligned}
\|p_2\mathcal{R}(\tilde{f}_0)\|_\infty &\leq c_{\mathcal{R}} \\
\|p_2\mathcal{N}(\delta)\|_\infty &\leq c_{\mathcal{N}}\|\delta\|^2 \\
\|p_2(\mathcal{N}(\delta) - \mathcal{N}(\gamma))\|_\infty &\leq c_{\mathcal{N}}(\|\delta\| + \|\gamma\|)\|\delta - \gamma\| \\
\|p_2(\tilde{\mathcal{L}} - \mathcal{L})\delta\|_\infty &\leq c_{\tilde{\mathcal{L}}}\|\delta\|
\end{aligned}$$

with constants $c_{\mathcal{R}} = 10^{-6}$, $c_{\mathcal{N}} = 4$ and $c_{\tilde{\mathcal{L}}} = 4 \cdot 10^{-5}$.

- (3) Combining these results we prove that $\tilde{\mathcal{K}}$ has a unique fixed point in X . This follows immediately from the estimates from points (1) and (2) and from the contraction mapping principle. Indeed, for all $\delta, \gamma \in X$ we have

$$\begin{aligned}\|\tilde{\mathcal{K}}(\delta)\| &\leq c_{\mathcal{L}}(c_{\mathcal{R}} + c_{\tilde{\mathcal{L}}}\|\delta\| + c_{\mathcal{N}}\|\delta\|^2) \\ &\leq 180(10^{-6} + (4 \cdot 10^{-5}) \cdot (5 \cdot 10^{-4}) + 4 \cdot (5 \cdot 10^{-4})^2) \\ &= 3.636 \cdot 10^{-4} < 5 \cdot 10^{-4},\end{aligned}$$

so $\tilde{\mathcal{K}}$ maps back into X , and $\tilde{\mathcal{K}}$ is a contraction because

$$\begin{aligned}\|\tilde{\mathcal{K}}(\delta) - \tilde{\mathcal{K}}(\gamma)\| &= \|\tilde{\mathcal{L}}^{-1}(\mathcal{N}(\delta) - \mathcal{N}(\gamma) + (\tilde{\mathcal{L}} - \mathcal{L})(\delta - \gamma))\| \\ &\leq c_{\mathcal{L}}(c_{\mathcal{N}}(\|\delta\| + \|\gamma\|) + c_{\tilde{\mathcal{L}}}\|\delta - \gamma\|) \\ &\leq 180(4 \cdot 10^{-3} + 4 \cdot 10^{-5})\|\delta - \gamma\| \\ &= 0.7272\|\delta - \gamma\|.\end{aligned}$$

Consequently, by the contraction mapping principle and elementary regularity theory there exists a $\delta_0 \in X \cap C^\infty([0, \infty))$ that solves (2.2). But such a δ_0 must also solve (2.1) and thus, $f_0 := \tilde{f}_0 + \delta$ must solve (1.2) and $f_0 \in C^\infty([0, \infty))$.

Remark 2.1. The main difficulty in the above procedure is to determine the approximate solution \tilde{f}_0 and the operator $\tilde{\mathcal{L}}$ such that the constants $c_{\mathcal{R}}$ and $c_{\tilde{\mathcal{L}}}$ are small enough for $\tilde{\mathcal{K}}$ to be a contraction. In contrast, we do not have much influence on the constants $c_{\mathcal{N}}$ and $c_{\mathcal{L}}$; they are the constants of our problem.

Remark 2.2. Because of the complicated form of the approximations \tilde{f}_0 and $\tilde{\mathcal{L}}$, our proof relies on computer algebra and rigorous computer-assisted methods for estimating rational functions (namely the method of interval arithmetic). These methods will be described in detail in the following sections.

2.1. Estimate for the remainder term $\mathcal{R}(\tilde{f}_0)$. The remainder term is the simplest one to analyze so we shall use it to demonstrate the method of interval arithmetic, which we shall use extensively throughout the paper. We defined the remainder term as

$$\mathcal{R}(\tilde{f}_0) = \tilde{f}_0'' + \left(\frac{2}{y} - \frac{y}{2}\right) \tilde{f}_0' - \frac{1}{y^2} \sin(2\tilde{f}_0)$$

with \tilde{f}_0 given explicitly as

$$\tilde{f}_0(y) = 2 \arctan(g_0(y)), \quad g_0(y) = \sum_{n=0}^{14} (f_0)_n T_{2n+1} \left(\frac{y}{\sqrt{2+y^2}} \right).$$

Now we argue that the remainder term, multiplied by the weight p_2 , is a rational function of y (we will need this fact later on to apply the interval arithmetic bounds). The main difficulty lies in convincing oneself that the square root in the definition of g_0 eventually does not show up in the expression $p_2 \mathcal{R}(\tilde{f}_0)$.

Let us start by writing the remainder term in terms of g_0

$$\mathcal{R}(\tilde{f}_0) = \frac{2}{1+g_0^2} \left(g_0'' + \left(\frac{2}{y} - \frac{y}{2} - \frac{2g_0 g_0'}{1+g_0^2} \right) g_0' - \frac{2g_0(1-g_0^2)}{y^2(1+g_0^2)} \right). \quad (2.3)$$

Now, because $T_{2n+1}(x) = xP_n(x^2)$, with P_n being some polynomial of order n , we can write the function g_0 as $g_0(y) = \frac{y}{\sqrt{2+y^2}}Q(y^2/(2+y^2))$ (with some polynomial Q). Consequently, we factored out the square root. It is now straightforward to see that the square root can be eventually factored out of $\mathcal{R}(\tilde{f}_0)$ in a similar fashion. Going back to the definition of the weight p_2 we obtain the representation

$$p_2\mathcal{R}(\tilde{f}_0)(y) = \frac{U(y^2/(2+y^2))}{V(y^2/(2+y^2))}$$

where U and V are polynomials with rational coefficients. As the last step, we compactify the domain to the interval $[0, 1]$ by replacing $y \in [0, \infty)$ with $x = y^2/(2+y^2) \in [0, 1]$. We refrain from writing down the polynomials U and V explicitly as they are of order 44 and 59, respectively, with large integer coefficients.

We now run into the core of the problem: how to estimate such a complicated rational function? Depending on how rigorous we want to be, such an estimate might be straightforward and produced by simply plotting the graph of the function or incredibly difficult if we decide to work on the rational function directly and show the bound explicitly. We decided on an approach that is almost as simple as plotting the function but still rigorous: interval arithmetic.

Interval arithmetic is essentially a way to find bounds on the range of a function on a given interval. Say we are interested in estimating the range of the function $f(x) = x - x^2$ on an interval $x \in [0, 1]$ (naturally, one can do this explicitly but the point is to illustrate the method). In the interval arithmetic approach we first compute the range of x , which is $[0, 1]$, then we compute the range of $-x^2$, which is $[-1, 0]$. Now the key observation is that the range of a sum has to be contained in the sum of the ranges (defined as $[a, b] + [c, d] := [a+c, b+d]$), that is

$$f(x) \in [0, 1] + [-1, 0] = [-1, 1], \quad x \in [0, 1].$$

Indeed, one finds that the exact range of the function f is $[0, 1/4] \subset [-1, 1]$.

One can define the remaining operations on intervals as follows.

Definition 2.3 (Interval arithmetic). For $a, b, c, d \in \mathbb{R}$ we define

$$\begin{aligned} [a, b] + [c, d] &:= [a+c, b+d] \\ [a, b] - [c, d] &:= [a-d, b-c] \\ [a, b] \cdot [c, d] &:= [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \\ \frac{[a, b]}{[c, d]} &:= [a, b] \cdot \left[\frac{1}{d}, \frac{1}{c}\right], \quad 0 \notin [c, d] \end{aligned}$$

(here we assume that the intervals on the left hand side are nonempty, so that $a \leq b$ and $c \leq d$). Moreover, the operations mixing intervals and real numbers can be included by interpreting $a \in \mathbb{R}$ as $[a, a]$.

It is straightforward to check that these definitions lead to the following statement.

Theorem 2.4. *Let $*$ be any of the operations defined in 2.3 and let $x \in [a, b]$ and $y \in [c, d]$. Then we have $x * y \in [a, b] * [c, d]$.*

This theorem allows us to extend the operations on numbers, like in our example $x - x^2$, to operations on sets. Consequently, $f(x) = x - x^2$ can be interpreted as either a function on real numbers giving $f(1) = 0$, or a function on intervals giving $f([0, 1]) = [-1, 1]$. Thanks to Theorem 2.4, for every $x \in [0, 1]$, we have $f(x) \in f([0, 1]) = [-1, 1]$.

There are, however, some pitfalls one should be aware of when using interval arithmetic. The first problem (also called the dependency problem) is that the resulting bound strongly depends on the algebraic form of the expression. For example, if we write $f(x)$ as $x(1 - x)$ the bound becomes $f([0, 1]) = [0, 1]$, so rewriting the expression might improve or degrade the estimate; the resulting bound will still be rigorous but it may simply be less efficient. Next, in most cases a single interval is insufficient to obtain a satisfactory estimate and splitting the interval into two or more subintervals will often improve the result. For example we can write the interval $[0, 1]$ as $[0, 1/2] \cup [1/2, 1]$, effectively splitting the domain into two parts; so for $x \in [0, 1]$ we have $f(x) = x(1 - x) \in f([0, 1/2]) \cup f([1/2, 1]) = [0, 1/2] \cup [0, 1/2] = [0, 1/2]$, which is closer to the optimal estimate. For a comprehensive summary of these and other aspects of interval arithmetic methods, the reader is referred to [23] or [24].

We remove the ambiguity coming from the dependency problem by writing each rational function R , which we want to estimate on the domain $x \in [0, 1]$, in so-called Bernstein form

$$R(x) = \frac{\sum_{i=0}^n a_i x^i (1 - x)^{n-i}}{\sum_{j=0}^m b_j x^j (1 - x)^{m-j}}$$

where $a_i, b_j \in \mathbb{Q}$. This representation is unique provided the fraction is reduced. The Bernstein form seems to improve the estimates coming from interval arithmetic (note that writing $f(x)$ from our example as $x(1 - x)$ is actually rewriting it in Bernstein form). Another benefit of the Bernstein form is that it is trivial to see if the denominator of R is strictly positive by simply checking if all the coefficients b_j are nonnegative (still, at least one has to be positive).

Then, we mince the domain of our function by bisecting each interval $[a, b]$, for which the interval $R([a, b])$ turned out to be too broad, and then we take a union of the resulting estimates. Specifically, we rewrite the offending interval as $[a, b] = [a, (a + b)/2] \cup [(a + b)/2, b]$ and we use $R([a, b]) \subset R([a, (a + b)/2]) \cup R([(a + b)/2, b])$.

Unfortunately, the rational functions we are dealing with are too complex and the number of subintervals too large to perform all the computations by hand or even explicitly include them in this paper. However, by following the procedure just explained, it is straightforward to verify our claims using any suitable software package. In addition, there exists an on-line supplement to this article which consists of a Mathematica Notebook that contains all the computations. Also, having discussed the method in detail here, in the rest of this paper we shall use the interval arithmetic freely.

Running the described algorithm we get

$$\|p_2 \mathcal{R}(\tilde{f}_0)\|_\infty \leq c_{\mathcal{R}}, \quad c_{\mathcal{R}} = 10^{-6}.$$

2.2. Estimates for the nonlinear term $\mathcal{N}(\delta)$. The nonlinear term $\mathcal{N}(\delta)$ was defined as

$$\begin{aligned} \mathcal{N}(\delta) &= \frac{1}{y^2} (2 \cos(2\tilde{f}_0) \delta - 2 \cos(2\tilde{f}_0 + \delta) \sin(\delta)) \\ &= \frac{2}{y^2} (\cos(2\tilde{f}_0) (\delta - \sin(\delta)) + 4 \cos(\delta/2) \sin(2\tilde{f}_0 + \delta/2) \sin^2(\delta/2)). \end{aligned}$$

so that

$$|\mathcal{N}(\delta)| \leq \frac{2}{y^2} \left(\frac{1}{6} |\delta|^3 + |\delta|^2 \left(|\sin(2\tilde{f}_0)| + \frac{1}{2} |\delta| \right) \right) = \frac{2}{y^2} |\delta|^2 \left(|\sin(2\tilde{f}_0)| + \frac{2}{3} |\delta| \right).$$

This leads to

$$|p_2 \mathcal{N}(\delta)| \leq (p_1 \delta)^2 \left(\left| \frac{2p_2}{p_1^2 y^2} \sin(2\tilde{f}_0) \right| + \frac{4p_2}{3p_1^3 y^2} |p_1 \delta| \right)$$

and now, according to the previous section, interval arithmetic provides the following bound on the first term,

$$\left| \frac{2p_2}{p_1^2 y^2} \sin(2\tilde{f}_0) \right| \leq 3.9, \quad (2.4)$$

while the second term can be bounded explicitly using the definitions of the weights p_1 , p_2 and p_3 ,

$$\frac{4p_2}{3p_1^3 y^2} = \frac{8(2 + y^2)}{9(4 + y^2)} \leq 1. \quad (2.5)$$

In summary, we obtain

$$\|p_2 \mathcal{N}(\delta)\|_\infty \leq \|\delta\|^2 (3.9 + \|\delta\|).$$

Now, if $\delta \in X$, we have $\|\delta\| \leq 5 \cdot 10^{-4}$ so that

$$\|p_2 \mathcal{N}(\delta)\|_\infty \leq c_{\mathcal{N}} \|\delta\|^2, \quad c_{\mathcal{N}} = 4.$$

A similar reformulation can be carried out for the difference

$$\begin{aligned} \mathcal{N}(\delta) - \mathcal{N}(\gamma) &= \frac{2}{y^2} (\cos(2\tilde{f}_0)(\delta - \gamma) - \cos(2\tilde{f}_0 + \delta) \sin(\delta) + \cos(2\tilde{f}_0 + \gamma) \sin(\gamma)) \\ &= \frac{2}{y^2} (2 \sin((\delta + \gamma)/2) \sin(2\tilde{f}_0 + (\delta + \gamma)/2) \sin(\delta - \gamma) \\ &\quad + \cos(2\tilde{f}_0)((\delta - \gamma) - \sin(\delta - \gamma))), \end{aligned}$$

Given the new form of the difference we can apply essentially the same estimates as for the term $\mathcal{N}(\delta)$ leading to

$$|p_2(\mathcal{N}(\delta) - \mathcal{N}(\gamma))| \leq |p_1(\delta - \gamma)|(|p_1 \delta| + |p_1 \gamma|) \left(\frac{2p_2}{y^2 p_1^2} |\sin(2\tilde{f}_0)| + \frac{4p_2}{3y^2 p_1^3} (|p_1 \delta| + |p_1 \gamma|) \right)$$

and then, again using (2.4) and (2.5), to

$$\|p_2(\mathcal{N}(\delta) - \mathcal{N}(\gamma))\|_\infty \leq \|\delta - \gamma\|(\|\delta\| + \|\gamma\|)(3.9 + \|\delta\| + \|\gamma\|).$$

But for $\delta, \gamma \in X$ we have $\|\delta\| + \|\gamma\| \leq 2 \cdot 5 \cdot 10^{-4}$ so

$$\|p_2(\mathcal{N}(\delta) - \mathcal{N}(\gamma))\|_\infty \leq c_{\mathcal{N}} \|\delta - \gamma\|(\|\delta\| + \|\gamma\|).$$

2.3. **The approximate operator $\tilde{\mathcal{L}}$.** We can write an inverse of \mathcal{L} as

$$\mathcal{L}^{-1}\delta(y) = \int_0^\infty G(x, y)\delta(x) dx$$

with the Green's function defined as

$$G(x, y) = \frac{-1}{W(w_0, w_1)(x)} \begin{cases} w_0(y)w_1(x) & y \leq x \\ w_0(x)w_1(y) & x \leq y \end{cases}$$

where $W(w_0, w_1) = w_0w_1' - w_0'w_1$. The functions w_0 and w_1 satisfy the differential equation

$$\mathcal{L}w_0 = \mathcal{L}w_1 = 0$$

with boundary conditions

$$\begin{aligned} w_0(y) &= y + \mathcal{O}(y^3), & y &\rightarrow 0 \\ w_1(y) &= 1 + \mathcal{O}(y^{-2}), & y &\rightarrow \infty. \end{aligned} \tag{2.6}$$

Naturally, these fundamental solutions are unknown in closed form.

Assume though, that we can find a pair of approximate solutions $\tilde{w}_0 \approx w_0$ and $\tilde{w}_1 \approx w_1$ with the same boundary conditions as the exact solutions. Then the solutions $(\tilde{w}_0, \tilde{w}_1)$ uniquely determine an operator $\tilde{\mathcal{L}} = \mathcal{L} + P\partial_y + Q$ by demanding that

$$\tilde{\mathcal{L}}\tilde{w}_0 = \tilde{\mathcal{L}}\tilde{w}_1 = 0.$$

Under such conditions, the coefficients P and Q are given by

$$\begin{aligned} P &= \frac{-1}{W(\tilde{w}_0, \tilde{w}_1)} (\mathcal{L}\tilde{w}_1\tilde{w}_0 - \mathcal{L}\tilde{w}_0\tilde{w}_1), \\ Q &= \frac{1}{W(\tilde{w}_0, \tilde{w}_1)} (\mathcal{L}\tilde{w}_1\tilde{w}_0' - \mathcal{L}\tilde{w}_0\tilde{w}_1'). \end{aligned}$$

With \tilde{w}_0 and \tilde{w}_1 known explicitly, the operator $\tilde{\mathcal{L}}$ has a closed form Green's function

$$\tilde{G}(x, y) = \frac{-1}{W(\tilde{w}_0, \tilde{w}_1)(x)} \begin{cases} \tilde{w}_0(y)\tilde{w}_1(x) & y \leq x \\ \tilde{w}_0(x)\tilde{w}_1(y) & x \leq y \end{cases}.$$

Now let us compute a robust, even if slightly inefficient, estimate on $c_{\mathcal{L}}$. For that we are going to need the free part \mathcal{L}_0 of the operator \mathcal{L} , defined as

$$\mathcal{L}_0\delta(y) = -\delta''(y) - \left(\frac{2}{y} - \frac{y}{2}\right)\delta'(y) + \frac{2}{y^2}\delta(y),$$

which is simply \mathcal{L} without the potential term \tilde{V}_0 . Let us denote the corresponding free Green's function by

$$G_0(x, y) = \frac{-1}{W(v_0, v_1)(x)} \begin{cases} v_0(y)v_1(x) & y \leq x \\ v_0(x)v_1(y) & x \leq y \end{cases}, \tag{2.7}$$

where v_0 and v_1 are fundamental solutions of \mathcal{L}_0 with the same boundary behavior as w_0 and w_1 (condition (2.6)). In fact, these solutions are known explicitly,

$$v_0(y) = \frac{3}{y^2}e^{y^2/4}(-y + (2 + y^2)D_+(y/2)), \quad v_1(y) = 1 + \frac{2}{y^2}$$

where D_+ is the Dawson integral

$$D_+(y) = e^{-y^2} \int_0^y e^{x^2} dx.$$

It is not hard to see that $v_0, v_1 > 0$ on $(0, \infty)$ and the Wronskian is simply

$$\frac{-1}{W(v_0, v_1)(x)} = \frac{1}{6} x^2 e^{-\frac{x^2}{4}}. \quad (2.8)$$

Now we are ready to take a closer look at the inverse of $\tilde{\mathcal{L}}$. We start by rewriting

$$\begin{aligned} p_1(y) \tilde{\mathcal{L}}^{-1} \alpha(y) &= \int_0^\infty \tilde{G}(x, y) p_1(y) \alpha(x) dx \\ &= \int_0^\infty \left(G_0(x, y) \frac{p_1(y)}{p_2(x)} \right) \left(\frac{\tilde{G}(x, y)}{G_0(x, y)} \right) p_2(x) \alpha(x) dx. \end{aligned} \quad (2.9)$$

For convenience let us denote the ratio \tilde{G}/G_0 as

$$H(x, y) := \frac{\tilde{G}(x, y)}{G_0(x, y)}.$$

Taking the $L^\infty(0, \infty)$ norm of (2.9), and pulling some terms out of the integral, we get

$$\begin{aligned} \|p_1 \tilde{\mathcal{L}}^{-1} \alpha\|_\infty &\leq \sup_{y>0} \left(\int_0^\infty |G_0(x, y)| \frac{p_1(y)}{p_2(x)} dx \right) \|H\|_\infty \|p_2 \alpha\|_\infty \\ &= \sup_{y>0} \left(\int_0^\infty G_0(x, y) \frac{p_1(y)}{p_2(x)} dx \right) \|H\|_\infty \|p_2 \alpha\|_\infty, \end{aligned}$$

where we abbreviate $\|H\|_\infty := \sup_{x, y>0} |H(x, y)|$. In the last equality we dropped the absolute value using the positivity of G_0 . By our choice of the weights p_1 and p_2 (this is actually their defining property), we have $\mathcal{L}_0 \frac{1}{p_1} = \frac{1}{p_2}$ or, equivalently,

$$\int_0^\infty G_0(x, y) \frac{p_1(y)}{p_2(x)} dx = 1 \quad (2.10)$$

hence,

$$\|p_1 \tilde{\mathcal{L}}^{-1} \alpha\|_\infty \leq \|H\|_\infty \|p_2 \alpha\|_\infty.$$

Similarly, for the derivative we have

$$p_3(y) (\tilde{\mathcal{L}}^{-1} \alpha)'(y) = \int_0^\infty \left(\partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} \right) H_1(x, y) p_2(x) \alpha(x) dx, \quad (2.11)$$

where

$$H_1(x, y) := \frac{\partial_y \tilde{G}(x, y)}{\partial_y G_0(x, y)}.$$

To estimate the above integral, we start with the following trick: we first take a derivative of (2.10) after dividing it by p_1 to get

$$\int_0^\infty \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} dx = 1, \quad \frac{1}{p_3(y)} = \left(\frac{1}{p_1} \right)'(y) = \frac{2\sqrt{2}}{(2+y^2)^{3/2}}. \quad (2.12)$$

The formula on the right above is, in fact, the defining property of the weight p_3 . Unfortunately we cannot use the elegant formula (2.12) directly, as we did with (2.10), because $\partial_y G_0(x, y)$ changes sign on the diagonal $\{(x, y) : x = y\}$. Indeed, if we go back to the definitions of fundamental solutions we see that $v'_0(y) = 12e^{y^2/4}y^{-3}(y/2 - D_+(y/2))$ is positive whereas $v'_1(y) = -4/y^3$ is negative. Consequently, we cannot remove the absolute value in the estimate

$$\|p_3(\tilde{\mathcal{L}}^{-1}\alpha)'\|_\infty \leq \sup_{y>0} \left(\int_0^\infty \left| \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} \right| dx \right) \|H_1\|_\infty \|p_2\alpha\|_\infty.$$

Thus, we have to take a brief detour to deal with the integral

$$\int_0^\infty \left| \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} \right| dx. \quad (2.13)$$

Luckily, this integral can be computed explicitly. To see this, we use the definition (2.7) of the Green's function to split (2.13) into two integrals

$$\begin{aligned} \int_0^\infty \left| \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} \right| dx &= -\frac{1}{6} \int_0^y x^2 e^{-\frac{x^2}{4}} v'_1(y) v_0(x) \frac{p_3(y)}{p_2(x)} dx \\ &\quad + \frac{1}{6} \int_y^\infty x^2 e^{-\frac{x^2}{4}} v_1(x) v'_0(y) \frac{p_3(y)}{p_2(x)} dx \\ &=: I_0(y) + I_1(y) \end{aligned}$$

Above we also used the fact that v_0 , v'_0 and v_1 , as well as the weights p_2 and p_3 , are all nonnegative and that v'_1 is negative. The integral $I_1(y)$ can be computed explicitly,

$$\begin{aligned} I_1(y) &= \frac{1}{6} v'_0(y) p_3(y) \int_y^\infty x^2 e^{-\frac{x^2}{4}} v_1(x) \frac{1}{p_2(x)} dx \\ &= \frac{1}{\sqrt{2}} v'_0(y) p_3(y) \int_y^\infty \frac{x(4+x^2)}{(2+x^2)^{3/2}} e^{-\frac{x^2}{4}} dx \\ &= \sqrt{2} v'_0(y) p_3(y) \frac{e^{-\frac{y^2}{4}}}{\sqrt{2+y^2}}. \end{aligned}$$

By definition, $p_3(y) = (2+y^2)^{3/2}/(2\sqrt{2})$ and thus, we arrive at the final formula

$$I_1(y) = 6y^{-3}(2+y^2)(y/2 - D_+(y/2)) = 3 - 2y^{-1}e^{-\frac{y^2}{4}}v_0(y).$$

In the last equality we used the definition of v_0 to replace D_+ with v_0 .

To compute I_0 , we go back to (2.12) and observe that

$$1 = \int_0^\infty \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} dx = -I_0(y) + I_1(y).$$

Consequently, $I_0(y) = I_1(y) - 1$ and thus,

$$\int_0^\infty \left| \partial_y G_0(x, y) \frac{p_3(y)}{p_2(x)} \right| dx = I_0(y) + I_1(y) = 2I_1(y) - 1 = 5 - 4y^{-1}e^{-\frac{y^2}{4}}v_0(y) \leq 5.$$

This yields

$$\|p_3(\tilde{\mathcal{L}}^{-1}\alpha)'\|_\infty \leq 5\|H_1\|_\infty\|p_2\alpha\|_\infty$$

and thus,

$$\begin{aligned}\|\tilde{\mathcal{L}}^{-1}\alpha\| &= \|p_1\tilde{\mathcal{L}}^{-1}\alpha\|_\infty + \|p_3(\tilde{\mathcal{L}}^{-1}\alpha)'\|_\infty \\ &\leq (\|H\|_\infty + 5\|H_1\|_\infty)\|p_2\alpha\|_\infty.\end{aligned}$$

To estimate $\|H\|_\infty$, we write

$$H(x, y) = \frac{W(v_0, v_1)(x)}{W(\tilde{w}_0, \tilde{w}_1)(x)} \begin{cases} h_0(y)h_1(x) & y \leq x \\ h_0(x)h_1(y) & x \leq y \end{cases}$$

where

$$h_0(x) = \frac{\tilde{w}_0(x)}{v_0(x)}, \quad h_1(x) = \frac{\tilde{w}_1(x)}{v_1(x)}.$$

At this point it remains to pick a particular pair $(\tilde{w}_0, \tilde{w}_1)$, approximating (w_0, w_1) , which leads to satisfactory estimates on $\tilde{\mathcal{L}} - \mathcal{L}$ (see section 2.4 below). Our numerically inspired guess is

$$\tilde{w}_0(y) := v_0(y) \sum_{n=0}^{44} (w_0)_n T_{2n} \left(\frac{y}{\sqrt{y^2 + 4}} \right) \quad (2.14)$$

$$\tilde{w}_1(y) := v_1(y) \sum_{n=0}^{37} (w_1)_n T_n \left(\frac{y-2}{y+2} \right) \quad (2.15)$$

with the coefficients $(w_0)_n$ and $(w_1)_n$ presented in Tables B.2 and B.3 respectively. A cautious reader will notice that the definition (2.15) is not complete: the last two coefficients in (2.15), $(w_1)_{36}$ and $(w_1)_{37}$, are missing from Table B.3. To remove this ambiguity we impose two additional conditions that fix the last two coefficients; we demand that

$$\left. \frac{d}{dy} \frac{\tilde{w}_1(y)}{v_1(y)} \right|_{y=0} = 0, \quad \left. \frac{d}{dy} \frac{\tilde{w}_1(y)}{v_1(y)} \right|_{y=\infty} = 0.$$

These two conditions ensure that \tilde{w}_1 has the correct endpoint behavior. Both coefficients $(w_1)_{36}$ and $(w_1)_{37}$ contribute only a correction of order 10^{-12} to the whole sum.

We have $\|h_j\|_\infty \leq 1.01$ for $j = 0, 1$ (see the appendix), so

$$\|H\|_\infty \leq 1.01^2 \left\| \frac{W(v_0, v_1)}{W(\tilde{w}_0, \tilde{w}_1)} \right\|_\infty.$$

The contents of the norm on the right hand side is a function of a single variable and thus it can be easily estimated by the method of interval arithmetic. After performing the computations we arrive at

$$\|H\|_\infty \leq 1.01^2 \cdot 28 < 30.$$

The function H_1 can be estimated in a similar fashion. Indeed, we have

$$H_1(x, y) = \frac{W(v_0, v_1)(x)}{W(\tilde{w}_0, \tilde{w}_1)(x)} \begin{cases} \frac{\tilde{w}'_0(y)}{v'_0(y)} h_1(x) & y \leq x \\ h_0(x) \frac{\tilde{w}'_1(y)}{v'_1(y)} & x < y. \end{cases}$$

Again, by interval arithmetic, we show that $\|\tilde{w}'_j/v'_j\|_\infty \leq 1.01$ for $j = 0, 1$ and then the same estimate as for $\|H\|_\infty$ follows,

$$\|H_1\|_\infty \leq 1.01^2 \cdot 28 < 30.$$

Putting everything together, we arrive at

$$\|\tilde{\mathcal{L}}^{-1}\alpha\| \leq (\|H\|_\infty + 5\|H_1\|_\infty)\|p_2\alpha\|_\infty \leq c_{\mathcal{L}}\|p_2\alpha\|_\infty, \quad c_{\mathcal{L}} := (1 + 5) \cdot 30 = 180.$$

2.4. Linear part and the constant $c_{\tilde{\mathcal{L}}}$. We have already constructed the operator $\tilde{\mathcal{L}}$ and its inverse, now it remains to show that the difference $\tilde{\mathcal{L}} - \mathcal{L}$ is small. To this end we estimate

$$p_2(\tilde{\mathcal{L}} - \mathcal{L})\delta = p_2(P\delta' + Q\delta)$$

where Q and P were given in the previous section as

$$\begin{aligned} P &= \frac{-1}{W(\tilde{w}_0, \tilde{w}_1)} (\mathcal{L}\tilde{w}_1\tilde{w}_0 - \mathcal{L}\tilde{w}_0\tilde{w}_1), \\ Q &= \frac{1}{W(\tilde{w}_0, \tilde{w}_1)} (\mathcal{L}\tilde{w}_1\tilde{w}'_0 - \mathcal{L}\tilde{w}_0\tilde{w}'_1), \end{aligned} \tag{2.16}$$

with \tilde{w}_0 and \tilde{w}_1 defined in (2.14) and (2.15), respectively. Note that (2.16) implies

$$\begin{aligned} \|p_2(\tilde{\mathcal{L}} - \mathcal{L})\delta\|_\infty &\leq \left(\left\| \frac{p_2}{p_3} P \right\|_\infty \|p_3\delta'\|_\infty + \left\| \frac{p_2}{p_1} Q \right\|_\infty \|p_1\delta\|_\infty \right) \\ &\leq \left(\left\| \frac{p_2}{p_3} P \right\|_\infty + \left\| \frac{p_2}{p_1} Q \right\|_\infty \right) \|\delta\|, \end{aligned}$$

so it is enough to estimate each of the two terms in the big parenthesis. With the help of interval arithmetic we find that they obey

$$\begin{aligned} \left\| \frac{p_2}{p_3} P \right\|_\infty &\leq c_P = 2 \cdot 10^{-5}, \\ \left\| \frac{p_2}{p_1} Q \right\|_\infty &\leq c_Q = 2 \cdot 10^{-5}. \end{aligned}$$

and therefore,

$$\|p_2(\tilde{\mathcal{L}} - \mathcal{L})\delta\|_\infty \leq c_{\tilde{\mathcal{L}}}\|\delta\|, \quad c_{\tilde{\mathcal{L}}} := c_P + c_Q = 4 \cdot 10^{-5}.$$

3. LINEAR STABILITY OF f_0

In this section we prove that f_0 is linearly stable, apart from the gauge mode, in the sense that the self-adjoint operator \mathcal{A}_0 , defined in (1.5), has no eigenvalues in the interval $(-\infty, 0]$ other than $\lambda_0 = -1$. We divide the proof into two steps. First, we prove that W_0 , defined as the solution of the initial value problem

$$\mathcal{L}W_\lambda = \lambda W_\lambda, \quad W_\lambda(0) = 0, \quad W'_\lambda(0) = 1 \tag{3.1}$$

with $\lambda = 0$, has exactly one zero on $(0, \infty)$ (note that W_0 coincides with the fundamental solution w_0 up to normalization). Then, we prove that $W_{\lambda_0}(y) := yf'_0(y)$ is an eigenfunction of \mathcal{A}_0 to the eigenvalue $\lambda_0 = -1$. Finally, we apply the Sturm oscillation theorem, which relates the number of zeros of W_0 on $(0, \infty)$ to the number of eigenvalues below zero. Since

W_0 has exactly one zero, there is only a single eigenvalue below zero. But this must be the eigenvalue $\lambda_0 = -1$ to the gauge mode W_{λ_0} .

3.1. Counting the zeros.

Lemma 3.1. *The solution W_0 of the initial value problem (3.1) with $\lambda = 0$ has exactly one zero in $(0, \infty)$. Moreover,*

$$q(y) := W_0(y)/y$$

is negative and decreasing for $y \geq 3$.

Proof. Let us note that the differential equation for W_0 can be written in terms of q as

$$q''(y) = \frac{1}{2y}(y^2 - 8) \left(q'(y) + \frac{q(y)}{y} \right) + \frac{4}{y^2} \cos(\tilde{f}_0(y))^2 q(y).$$

From the structure of the above equation one immediately notices that, if $q(y) < 0$ and $q'(y) < 0$ for some $y \geq 3$, then $q''(y) < 0$. Consequently, if $q(3) < 0$ and $q'(3) < 0$, it follows that

$$q(y) < 0, \quad q'(y) < 0 \quad \text{for all } y \geq 3. \quad (3.2)$$

The second observation is that if $q(0) > 0$, $q(3) < 0$ and $q'(y) < 0$ for all $y \in [0, 3]$ then the function q has exactly one zero in $[0, 3]$. In combination with (3.2) this implies that q has exactly one zero in $[0, \infty)$ and thus, W_0 has exactly one zero in $(0, \infty)$. We shall now show that these preconditions do indeed occur for q .

First let us denote $\delta_0 := W_0 - \tilde{w}_0$, where \tilde{w}_0 is the approximation to w_0 introduced in the previous section. Rewriting (3.1) for δ_0 we have

$$\mathcal{L}\delta_0 = -\mathcal{L}\tilde{w}_0, \quad \delta_0(0) = 0, \quad \delta'_0(0) = 1 - \tilde{w}'_0(0)$$

or, equivalently,

$$\tilde{\mathcal{L}}\delta_0 = -\mathcal{L}\tilde{w}_0 + (\tilde{\mathcal{L}} - \mathcal{L})\delta_0 \quad (3.3)$$

with the same initial condition. Note that one could easily normalize $\tilde{w}'_0(0)$ to one but this is not necessary for our line of reasoning. Eq. (3.3) can be treated with the machinery from the previous sections. However, before we proceed we need one more technical modification.

The main difference between (3.3) and (2.2) is that the source term $-\mathcal{L}\tilde{w}_0$ in (3.3) is unbounded and grows exponentially as $y \rightarrow \infty$ and we are therefore unable to apply the estimates on $\tilde{\mathcal{L}}^{-1}$ from the previous section directly. To fix this issue we regularize the source term $-\mathcal{L}\tilde{w}_0$ outside of $[0, 3]$ by multiplying it with the indicator function

$$\chi(y) = \begin{cases} 1 & y \in [0, 3] \\ 0 & y \in (3, \infty). \end{cases}$$

Let us now consider the solution $\hat{\delta}_0$ to the ad-hoc regularized boundary value problem

$$\begin{aligned} \tilde{\mathcal{L}}\delta &= -\chi\mathcal{L}\tilde{w}_0 + (\tilde{\mathcal{L}} - \mathcal{L})\delta, \\ \delta(0) &= 0, \quad \delta(\infty) = \text{const.} \end{aligned} \quad (3.4)$$

For the sake of the argument we will assume that the problem (3.4) can actually be solved (we prove this explicitly in the next paragraph). With the solution $\hat{\delta}_0$ at hand, we define a

function $\widehat{W}_0 := c_0(\tilde{w}_0 + \widehat{\delta}_0)$ with $c_0 = 1/(\tilde{w}'_0(0) + \widehat{\delta}'_0(0))$, where we postpone the verification that c_0 is finite for the sake of clarity. By construction, \widehat{W}_0 solves the initial value problem

$$\mathcal{L}\widehat{W}_0 = c_0(1 - \chi)\mathcal{L}\tilde{w}_0, \quad \widehat{W}_0(0) = 0, \quad \widehat{W}'_0(0) = 1. \quad (3.5)$$

Observe now that the right hand side of the differential equation in (3.5) is identically zero on $[0, 3]$. This means that W_0 and \widehat{W}_0 coincide on $[0, 3]$, i.e., $W_0 = \widehat{W}_0 = c_0(\tilde{w}_0 + \widehat{\delta}_0)$ on $[0, 3]$. But the expression $c_0(\tilde{w}_0 + \widehat{\delta}_0)$ is actually semi-explicit: we know \tilde{w}_0 explicitly and we will show that $\widehat{\delta}_0$ is small, which leads to precise pointwise bounds on W_0 .

To show that there exists a solution $\widehat{\delta}_0$, we proceed as in the previous sections: we solve Eq. (3.4) with a fixed point argument applied to the map

$$\mathcal{J}(\delta) := \tilde{\mathcal{L}}^{-1}(-\chi\mathcal{L}\tilde{w}_0 + (\tilde{\mathcal{L}} - \mathcal{L})\delta)$$

acting on the ball

$$Y = \{\delta \in C^1([0, \infty)) : \|\delta\| \leq 0.03\}.$$

Before we move on, let us take a closer look at the term $-\chi\mathcal{L}\tilde{w}_0$. We constructed \tilde{w}_0 so that $\tilde{\mathcal{L}}\tilde{w}_0 = 0$, thus

$$-\chi\mathcal{L}\tilde{w}_0 = \chi(\tilde{\mathcal{L}} - \mathcal{L})\tilde{w}_0 = \chi(P\tilde{w}'_0 + Q\tilde{w}_0) = (P\chi\tilde{w}'_0 + Q\chi\tilde{w}_0).$$

Using this identity we infer

$$\begin{aligned} \|p_2\chi\mathcal{L}\tilde{w}_0\|_\infty &= \|p_2(P\chi\tilde{w}'_0 + Q\chi\tilde{w}_0)\|_\infty \\ &\leq \left\|P\frac{p_2}{p_3}\right\|_\infty \|p_3\chi\tilde{w}'_0\|_\infty + \left\|Q\frac{p_2}{p_1}\right\|_\infty \|p_1\chi\tilde{w}_0\|_\infty \\ &\leq c_P\|p_3\tilde{w}'_0\|_{L^\infty(0,3)} + c_Q\|p_1\tilde{w}_0\|_{L^\infty(0,3)} \\ &= c_{\tilde{\mathcal{L}}} \left(\frac{1}{2}\|p_3\tilde{w}'_0\|_{L^\infty(0,3)} + \frac{1}{2}\|p_1\tilde{w}_0\|_{L^\infty(0,3)}\right) \\ &\leq c_{\tilde{\mathcal{L}}}c_{\tilde{w}_0}, \end{aligned}$$

where $c_{\tilde{w}_0}$ is a bound on the supremum, yet to be computed, and we made use of $c_P = c_Q = c_{\tilde{\mathcal{L}}}/2$ (cf. section 2.4). From this bound we immediately infer

$$\begin{aligned} \|\mathcal{J}(\delta)\| &\leq c_{\mathcal{L}}c_{\tilde{\mathcal{L}}}(c_{\tilde{w}_0} + \|\delta\|), \\ \|\mathcal{J}(\delta) - \mathcal{J}(\gamma)\| &\leq c_{\mathcal{L}}c_{\tilde{\mathcal{L}}}\|\delta - \gamma\|, \end{aligned} \quad (3.6)$$

for all $\delta, \gamma \in Y$, where $c_{\mathcal{L}}c_{\tilde{\mathcal{L}}} = 0.0072 < 1$. The only new estimate in (3.6) is the one for $c_{\tilde{w}_0}$, but it can be readily computed using interval arithmetic. We have³

$$\|p_3\tilde{w}'_0\|_{L^\infty(0,3)} \leq 1.2, \quad \|p_1\tilde{w}_0\|_{L^\infty(0,3)} \leq 4, \quad (3.7)$$

so that

$$c_{\tilde{w}_0} = \frac{1}{2}(1.2 + 4) = 2.6.$$

³ In the estimate (3.7) we are actually estimating a rational function times an exponential function. The interval arithmetic for rational functions, which we introduced earlier on via Definition 2.3 and Theorem 2.3, can be easily extended to include operations on exponents by defining $\exp([a, b]) := [\exp(a), \exp(b)]$ thanks to the monotonicity of \exp .

We conclude that the map \mathcal{J} maps Y into itself,

$$\|\mathcal{J}(\delta)\| \leq c_{\mathcal{L}}c_{\tilde{\mathcal{L}}}(2.6 + 0.03) \leq c_{\mathcal{L}}c_{\tilde{\mathcal{L}}} \cdot 3 = 0.0216 < 0.03$$

and thus, by the contraction mapping principle, we obtain the existence of $\widehat{\delta}_0 \in Y$ solving (3.4). The normalization constant $c_0 = 1/(\widetilde{w}'_0(0) + \widehat{\delta}'_0(0))$, used in the definition of \widehat{W}_0 , can now be easily verified to be finite and positive because, by explicit computation, $\widetilde{w}'_0(0) > 1$ and $|\widehat{\delta}'_0(0)| \leq \|\widehat{\delta}_0\| \leq 0.03 \ll 1$.

As already mentioned, we have $W_0 = \widehat{W}_0 = c_0(\widetilde{w}_0 + \widehat{\delta}_0)$ on $[0, 3]$, so the estimate

$$p_1|W_0/c_0 - \widetilde{w}_0| + p_3|W'_0/c_0 - \widetilde{w}'_0| \leq \|\widehat{\delta}_0\| \leq 0.03$$

holds on $[0, 3]$. In other words,

$$\begin{aligned} \widetilde{w}_0 - 0.03/p_1 &\leq W_0/c_0 \leq \widetilde{w}_0 + 0.03/p_1, \\ \widetilde{w}'_0 - 0.03/p_3 &\leq W'_0/c_0 \leq \widetilde{w}'_0 + 0.03/p_3 \end{aligned} \tag{3.8}$$

on $[0, 3]$. From (3.8) we infer

$$\begin{aligned} q(3) &= W_0(3)/3 \leq c_0(\widetilde{w}_0(3) + 0.03/p_1(3))/3 \leq -0.06c_0 < 0, \\ q'(3) &= W'_0(3)/3 - W_0(3)/3^2 \\ &\leq c_0(\widetilde{w}'_0(3) + 0.03/p_3(3))/3 - c_0(\widetilde{w}_0(3) - 0.03/p_1(3))/3^2 \leq -0.05c_0 < 0. \end{aligned}$$

Consequently, by our earlier result (3.2), q has no zeros in $[3, \infty)$, nor has W_0 . At the same time the interval arithmetic (with W_0 replaced by \widetilde{w}_0 according to the bounds (3.8)) reveals that

$$\begin{aligned} q(y) &> 0 && \text{for } y \in [0, 1], \\ q'(y) &< 0 && \text{for } y \in [1, 3] \end{aligned}$$

which, in combination with $q(3) < 0$, means that q traverses zero exactly once on $(0, \infty)$, and so does W_0 . \square

3.2. Applying the Sturm oscillation to \mathcal{A}_0 .

Lemma 3.2. *The operator \mathcal{A}_0 has no eigenvalue at $\lambda = 0$.*

Proof. We will show that $W_0 \notin \mathcal{H}$ and thus, W_0 is not an eigenfunction. The two possible behaviors of $W_0(y)$ when $y \rightarrow \infty$ are $W_0(y) = -1 + \mathcal{O}(y^{-2})$ or $W_0(y) = y^{-1}e^{y^{2/4}}(1 + \mathcal{O}(y^{-2}))$ (both are up to normalization). Only the first, recessive, behavior leads to $W_0 \in \mathcal{H}$. Assume that $W_0(y) = -1 + \mathcal{O}(y^{-2})$. Then $q(y) = -1/y + \mathcal{O}(y^{-3})$ and $q'(y) = 1/y^2 + \mathcal{O}(y^{-4})$, so $q'(y) > 0$ for sufficiently large y . But this is a contradiction to $q(y) < 0$ and $q'(y) < 0$ for $y \geq 3$ from Lemma 3.1. Thus, W_0 grows exponentially and $\lambda = 0$ is not an eigenvalue. \square

Lemma 3.3. *The function $W_{\lambda_0}(y) := yf'_0(y)$ is an eigenfunction of \mathcal{A}_0 to the eigenvalue $\lambda_0 = -1$. Furthermore, $W_{\lambda_0} > 0$ on $(0, \infty)$.*

Proof. One can easily check that if f_0 solves (1.2) then

$$W_{\lambda_0}(y) := yf'_0(y)$$

lies in \mathcal{H} and solves the differential equation $\mathcal{A}_0 W_{\lambda_0} = \lambda_0 W_{\lambda_0}$. Therefore, W_{λ_0} is an eigenfunction of \mathcal{A}_0 to the eigenvalue $\lambda_0 = -1$. Moreover, from Theorem 1.1 we know that f_0 is close to its approximation \tilde{f}_0 in the sense that

$$\tilde{f}_0'(y) - 5 \cdot 10^{-4}/p_3(y) \leq f_0'(y) \leq \tilde{f}_0'(y) + 5 \cdot 10^{-4}/p_3(y).$$

By applying this bound to the definition of W_{λ_0} we get

$$W_{\lambda_0}(y) \geq y \tilde{f}_0'(y) - 5 \cdot 10^{-4} y / p_3(y).$$

The right hand side can be written explicitly in the form

$$y \tilde{f}_0'(y) - 5 \cdot 10^{-4} y / p_3(y) = x(1 - x^2) \frac{\sum_{n=0}^N a_n x^{2n} (1 - x^2)^{N-n}}{\sum_{k=0}^K b_k x^{2k} (1 - x^2)^{K-k}}, \quad x = \frac{y}{\sqrt{2 + y^2}}$$

with $N, K = 59$ and all $a_n, b_k > 0$, ultimately yielding

$$W_{\lambda_0}(y) > 0, \quad y \in (0, \infty).$$

□

At this point, the proof of Theorem 1.6 reduces to an application of Sturm comparison and oscillation theorems. The comparison theorem is as follows.

Lemma 3.4. *Let W_λ be the unique solution of the initial value problem*

$$\mathcal{A}_0 W_\lambda = \lambda W_\lambda, \quad W_\lambda(0) = 0, \quad W_\lambda'(0) = 1.$$

If $\lambda \leq -1$ then W_λ has no zeros in $(0, \infty)$.

Proof. We follow the proof in [25], Theorem 2.6.3. The case $\lambda = -1$ is handled by Lemma 3.3, so assume $\lambda < \lambda_0 := -1$. From Lemma 3.3 we know that $W_{\lambda_0} > 0$ on $(0, \infty)$. Thus, $W_\lambda / W_{\lambda_0}$ is well-defined and a straightforward computation reveals the *Picone identity*

$$\left[\rho \frac{W_\lambda}{W_{\lambda_0}} (W_\lambda' W_{\lambda_0} - W_\lambda W_{\lambda_0}') \right]' = (\lambda_0 - \lambda) \rho W_\lambda^2 + \rho \frac{(W_\lambda' W_{\lambda_0} - W_\lambda W_{\lambda_0}')^2}{W_{\lambda_0}^2}$$

with $\rho(y) = y^2 e^{-y^2/4}$. Now assume that the statement is wrong and $y_0 > 0$ is the first zero of W_λ in $(0, \infty)$. Integrating the Picone identity from 0 to y_0 yields

$$0 = (\lambda_0 - \lambda) \int_0^{y_0} W_\lambda(y)^2 \rho(y) dy + \int_0^{y_0} \frac{[W_\lambda'(y) W_{\lambda_0}(y) - W_\lambda(y) W_{\lambda_0}'(y)]^2}{W_{\lambda_0}(y)^2} \rho(y) dy,$$

where we have used $\rho(0) = W_\lambda(y_0) = 0$. Consequently, since $\lambda_0 - \lambda > 0$, we infer $W_\lambda(y) = 0$ for all $y \in [0, y_0]$, which implies that W_λ is the trivial solution, a contradiction. □

Now the proof of Theorem 1.6 is readily completed by invoking Theorem 1.2 from [21]⁴, which states that in the interval $[\lambda, 0)$, $\lambda \leq -1$, there are exactly as many eigenvalues as there are zeros of W_0 minus the zeros of W_λ , that is, $1 - 0 = 1$. Consequently, there is exactly one eigenvalue in each interval $[\lambda, 0)$ for any $\lambda \leq -1$. This single eigenvalue has to be $\lambda_0 = -1$ from Lemma 3.3. Furthermore, we have already proved that W_0 is not an eigenfunction, so the only eigenvalue in $(-\infty, 0]$ is $\lambda_0 = -1$.

⁴Note that there is a small typo in Theorem 1.2 in [21]: $N(c)$ is supposed to denote the number of zeros of u_2 minus the number of zeros of u_1 , not the other way round.

3.3. Proof of Proposition 1.3. The proof is an extended version of an argument given in [3]. We write the equation (1.2) as

$$\frac{1}{\rho}(\rho f')' = S_1, \quad S_1(y) = \frac{\sin(2f(y))}{y^2} \quad (3.9)$$

Differentiating (3.9) $k - 1$ times we get

$$\left(\frac{1}{\rho}(\rho f')' \right)^{(k-1)} = S_1^{(k-1)}.$$

We then use the identity

$$\left(\frac{1}{\rho}(\rho f')' \right)'(y) - \frac{1}{\rho}(\rho f'')'(y) = - \left(\frac{2}{y^2} + \frac{1}{2} \right) f'(y)$$

to obtain a differential equation for f'' , which reads

$$\frac{1}{\rho}(\rho f'')'(y) = S_1'(y) + \left(\frac{2}{y^2} + \frac{1}{2} \right) f'(y) =: S_2(y).$$

By repeating this procedure, we find a differential equation for the k -th derivative of f ,

$$\frac{1}{\rho}(\rho f^{(k)})' = S_k, \quad S_k(y) = S_{k-1}'(y) + \left(\frac{2}{y^2} + \frac{1}{2} \right) f^{(k-1)}(y). \quad (3.10)$$

Multiplying (3.10) by ρ and integrating on $[y, \infty)$ gives us

$$\lim_{x \rightarrow \infty} \rho(x) f^{(k)}(x) - \rho(y) f^{(k)}(y) = \int_y^\infty \rho(x) S_k(x) dx$$

It is easy to see that any derivative of f will grow at most algebraically at infinity (this follows directly from (3.9) and its derivatives); at the same time $\rho(y) = y^2 e^{-\frac{y^2}{4}}$ decays exponentially at infinity so the limit on the left hand side is zero and the integral on the right hand side of (3.10) converges. We thus end up with

$$f^{(k)}(y) = \frac{- \int_y^\infty \rho(x) S_k(x) dx}{\rho(y)}.$$

Assume now that $|f^{(n)}(y)| \lesssim y^{-2-n}$ for any $n = 1, \dots, k-1$. Then it is easy to see that $|S_k(y)| \leq C_{k-1} y^{-1-k}$ for some constant $C_{k-1} > 0$ because, in the leading order, S_k consists of terms of the form $f^{(k-1)}(y)$ and $y^{-1-k} \sin(2f(y))$. In turn, this implies

$$|f^{(k)}(y)| \leq C_{k-1} \frac{\int_y^\infty \rho(x) x^{-1-k} dx}{\rho(y)} \leq C_{k-1} \frac{y^{-1-k} \int_y^\infty \rho(x) dx}{\rho(y)} \leq C_k y^{-2-k}.$$

Consequently, the claim $|f^{(k)}(y)| \leq C_k y^{-2-k}$ for all $k \in \mathbb{N}$ follows inductively.

The statement $f^{(2k)}(0) = 0$ for all $k \in \mathbb{N}_0$ follows essentially from the reflection symmetry of Eq. (1.2) under $y \mapsto -y$ and is easily proved inductively; we omit the details.

APPENDIX A. ESTIMATES ON THE RATIOS OF FUNDAMENTAL SOLUTIONS

The proof that the ratios $\|\tilde{w}_1/v_1\| \leq 1.01$ and $\|\tilde{w}'_1/v'_1\| \leq 1.01$ is a straightforward application of interval arithmetic as both w_1 and v_1 are rational functions. Things get more complicated when we try to compare \tilde{w}_0 and v_0 or \tilde{w}'_0 and v'_0 because, by definition,

$$v_0(y) = \frac{3}{y^2} e^{y^2/4} (-y + (2 + y^2) D_+(y/2))$$

contains Dawson's integral. As far as we know there is no definition of interval arithmetic for Dawson's integral, so we will have to resort to a more analytical approach. The goal now is to find an approximation \tilde{v}_0 to v_0 such that the ratio \tilde{w}_0/\tilde{v}_0 is rational, so that we can apply our interval arithmetic algorithm.

Let us remind that the Wronskian of v_0 and v_1 is

$$v_0 v'_1 - v'_0 v_1 = W(v_0, v_1) = -6y^{-2} e^{y^2/4}, \quad (\text{A.1})$$

or equivalently

$$\left(\frac{v_0}{v_1}\right)' - \frac{6e^{y^2/4}}{v_1(y)^2 y^2} = 0, \quad (\text{A.2})$$

so v_0 can be written in the form of the following integral

$$v_0(y) = 6v_1(y) \int_0^y \frac{e^{x^2/4}}{v_1(x)^2 x^2} dx.$$

Let us assume that we found a suitable approximation \tilde{v}_0 such that

$$\left(\frac{\tilde{v}_0}{v_1}\right)' - \frac{6e^{y^2/4}}{v_1(y)^2 y^2} = -\frac{6e^{y^2/4} \varepsilon(y)}{v_1(y)^2 y^2} \quad (\text{A.3})$$

with

$$0 \leq \varepsilon(y) \leq c_\varepsilon \ll 1. \quad (\text{A.4})$$

The integral representation of \tilde{v}_0 is simply

$$\begin{aligned} \tilde{v}_0(y) &= 6v_1(y) \int_0^y \frac{e^{x^2/4}}{v_1(x)^2 x^2} (1 - \varepsilon(x)) dx \\ &= v_0(y) - 6v_1(y) \int_0^y \frac{e^{x^2/4} \varepsilon(x)}{v_1(x)^2 x^2} dx. \end{aligned}$$

which, after taking into account (A.4), leads to

$$(1 - c_\varepsilon) v_0(y) \leq \tilde{v}_0(y) \leq v_0(y). \quad (\text{A.5})$$

This means that finding \tilde{v}_0 such that ε is small in the sense of (A.4) can be directly translated to a pointwise bound on v_0 . As for the estimate on the derivative v'_0 we subtract (A.2) from (A.3) and get

$$\tilde{v}'_0 - v'_0 = -\frac{6e^{y^2/4}}{v_1(y)^2 y^2} \varepsilon(y) + \frac{v'_1(y)}{v_1(y)} (\tilde{v}_0 - v_0)$$

The first term on the right hand side is negative and applying (A.5) to the second term (and remembering that $v'_1(y)/v_1(y) = -4/(2y + y^3) < 0$) we get

$$\tilde{v}'_0 - v'_0 \leq \frac{-c_\varepsilon}{1 - c_\varepsilon} \frac{v'_1}{v_1} \tilde{v}_0$$

which implies

$$\frac{\tilde{v}'_0(y)}{v'_0(y)} \leq \frac{1}{1 + \frac{c_\varepsilon}{1 - c_\varepsilon} \frac{v'_1(y)}{v_1(y)} \frac{\tilde{v}_0(y)}{\tilde{v}'_0(y)}}. \quad (\text{A.6})$$

Note that $v'_0, \tilde{v}'_0 \geq 0$.

To find the right approximation \tilde{v}_0 , we simply use a well established expression of D_+ as a continued fraction ([16], formula (2.7)),

$$D_+(z) = \frac{z}{1 + 2z^2 -} \frac{4z^2}{3 + 2z^2 -} \frac{8z^2}{5 + 2z^2 -} \frac{12z^2}{7 + 2z^2 -} \dots, \quad (\text{A.7})$$

and plug it directly into the definition of v_0 . Truncating (A.7) at the twelfth term we find, via interval arithmetic, that the associated \tilde{v}_0 has

$$0 \leq \varepsilon(y) \leq c_\varepsilon = \frac{1}{500}$$

so from (A.5) we have

$$\frac{\tilde{w}_0(y)}{v_0(y)} \leq \frac{\tilde{w}_0(y)}{\tilde{v}_0(y)}. \quad (\text{A.8})$$

As for the ratio of derivatives we use (A.6) to get

$$\frac{\tilde{w}'_0(y)}{v'_0(y)} \leq \frac{\tilde{w}'_0(y)}{\tilde{v}'_0(y)} \cdot \frac{1}{1 + \frac{c_\varepsilon}{1 - c_\varepsilon} \frac{v'_1(y)}{v_1(y)} \frac{\tilde{v}_0(y)}{\tilde{v}'_0(y)}}. \quad (\text{A.9})$$

The functions on the right hand sides of (A.8) and (A.9) are now explicit rational functions and they can be easily estimated by interval arithmetic which yields

$$\frac{\tilde{w}_0(y)}{v_0(y)} \leq 1.01, \quad \frac{\tilde{w}'_0(y)}{v'_0(y)} \leq 1.01.$$

Since all functions with subscript 0 are nonnegative, the claimed bounds follow.

APPENDIX B. TABLES OF COEFFICIENTS FOR APPROXIMATE SOLUTIONS

TABLE B.1. The coefficients of the approximate solution \tilde{f}_0

n	$(f_0)_n$	n	$(f_0)_n$
0	$\frac{268245}{72878}$	8	$\frac{1}{204079}$
1	$\frac{-3174}{105551}$	9	$\frac{1}{675805}$
2	$\frac{1897}{97022}$	10	$\frac{1}{1400761}$
3	$\frac{14}{72731}$	11	$\frac{1}{3586839}$
4	$\frac{79}{119383}$	12	$\frac{1}{7289041}$
5	$\frac{4}{66337}$	13	$\frac{1}{16940631}$
6	$\frac{5}{109368}$	14	$\frac{1}{59286294}$
7	$\frac{1}{109045}$		

TABLE B.2. The coefficients of \tilde{w}_0

n	$(w_0)_n$	n	$(w_0)_n$	n	$(w_0)_n$	n	$(w_0)_n$
0	$\frac{18741}{112373}$	1	$\frac{-61989}{170650}$	2	$\frac{2353}{10197}$	3	$\frac{-9791}{92415}$
4	$\frac{2796}{44407}$	5	$\frac{-1970}{54159}$	6	$\frac{723}{52459}$	7	$\frac{-1268}{126121}$
8	$\frac{1160}{216371}$	9	$\frac{-52}{33393}$	10	$\frac{113}{60053}$	11	$\frac{-66}{114007}$
12	$\frac{21}{98015}$	13	$\frac{-19}{49629}$	14	$\frac{-2}{77919}$	15	$\frac{-4}{43893}$
16	$\frac{2}{43899}$	17	$\frac{1}{42980}$	18	$\frac{4}{101127}$	19	$\frac{1}{77370}$
20	$\frac{1}{192093}$	21	$\frac{-1}{169116}$	22	$\frac{-1}{153707}$	23	$\frac{-1}{165215}$
24	$\frac{-1}{336932}$	25	$\frac{-1}{1215317}$	26	$\frac{1}{1285791}$	27	$\frac{1}{802012}$
28	$\frac{1}{855284}$	29	$\frac{1}{1353569}$	30	$\frac{1}{3240550}$	31	$\frac{-1}{47007496}$
32	$\frac{-1}{5407794}$	33	$\frac{-1}{4477156}$	34	$\frac{-1}{5647023}$	35	$\frac{-1}{9667418}$
36	$\frac{-1}{32837828}$	37	$\frac{1}{54537641}$	38	$\frac{1}{22028453}$	39	$\frac{1}{20327064}$
40	$\frac{1}{22814866}$	41	$\frac{1}{33327962}$	42	$\frac{1}{50263835}$	43	$\frac{1}{112134837}$
44	$\frac{1}{190131191}$						

TABLE B.3. The coefficients of \tilde{w}_1

n	$(w_1)_n$	n	$(w_1)_n$	n	$(w_1)_n$	n	$(w_1)_n$
0	$\frac{4607589}{9727120}$	1	$\frac{1737631}{2734940}$	2	$\frac{-3983}{1039272}$	3	$\frac{-256739}{1728298}$
4	$\frac{18231}{688103}$	5	$\frac{212793}{6236542}$	6	$\frac{-66549}{3547063}$	7	$\frac{-1981}{1692925}$
8	$\frac{13293}{2855801}$	9	$\frac{-10983}{6516796}$	10	$\frac{-28}{197979}$	11	$\frac{1701}{5019274}$
12	$\frac{-525}{3869749}$	13	$\frac{147}{5955391}$	14	$\frac{3}{664118}$	15	$\frac{-63}{6270472}$
16	$\frac{21}{2638831}$	17	$\frac{-7}{2304718}$	18	$\frac{-7}{18723692}$	19	$\frac{7}{6169514}$
20	$\frac{-21}{34570120}$	21	$\frac{7}{84719934}$	22	$\frac{3}{31279664}$	23	$\frac{21}{272250212}$
24	$\frac{3}{117220480}$	25	$\frac{21}{13759709834}$	26	$\frac{-21}{2997459842}$	27	$\frac{7}{1630774626}$
28	$\frac{-7}{6156689032}$	29	$\frac{-21}{64552514338}$	30	$\frac{21}{41777614736}$	31	$\frac{-21}{84914413922}$
32	$\frac{21}{547142712584}$	33	$\frac{3}{84895626842}$	34	$\frac{-3}{87634052792}$	35	$\frac{1}{80692012804}$

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